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Dynamical equations for a Regge theory with crossing symmetry and unitarity. IV. Coupled channels

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Integral equations for construction of a crossing-symmetric unitary Regge theory are extended to allow two coupled two-body channels. As in the case of a single channel, spectral functions are represented as Watson-Sommerfeld integrals over continued partial waves. A new type of partial wave is needed to represent one of the spectral functions in a region where one channel is open and the other is closed. This leads to certain difficulties in allowing realistic Regge poles.

I. INTRODUCTION

Integral equations for construction of a crossing-symmetric unitary Regge theory were described in the first papers of this series.[1-3] The equations were stated for meson-meson scattering, in the case of one neutral pseudoscalar meson. Unitarity is exact in the elastic region, while in the inelastic region there are absorptive effects of multiperipheral type arising from crossed two-particle processes, plus any additional inelasticity that may be introduced through an arbitrary central spectral function. Given a solution of the equations, an amplitude may be constructed which would have Mandelstam analyticity with correct support of spectral functions and exact crossing symmetry.

Extension of the equations to include the three isospin amplitudes of $\pi\pi$ scattering is straightforward. A more difficult problem is to allow several coupled two-body channels such that the two-body unitarity condition is not diagonal in an angular momentum and isospin basis. This is the problem discussed in the following for the case of two coupled channels called $\pi\pi$ and MM. The formalism is constructed in close analogy to that of Ref. 1 in that spectral functions are represented by Watson-Sommerfeld integrals over continued partial-wave amplitudes.[4]

There is no difficulty in constructing the spectral functions so as to ensure two-body unitarity, exact crossing symmetry, and proper support. A new feature is encountered, however, in representing the spectral function associated with $\pi\pi \to MM$ in the energy region where the MM channel is closed but $\pi\pi$ is open. It must be represented by a modified partial-wave amplitude which is not the usual Froissart-Gribov amplitude analytically continued from the region above the MM threshold. This complicates the use of partial-wave dispersion relations and the N/D method, so that the method of treating Regge poles developed in Ref. 2 is no longer effective as it stands. Until certain delicate problems of analytic continuation are solved, the present formalism gives a well-defined set of integral equations only when there are no Regge poles in the right-hand half of the angular momentum plane.

Section II summarizes kinematics, crossing conditions, and the form of the Mandelstam representations. Here, and throughout the paper, dispersion relations are written in unsubtracted form, as is appropriate to the case without Regge poles in the right-hand half plane.[1]

Section III introduces the Froissart-Gribov amplitudes.

In Sec. IV and V, the Watson-Sommerfeld representations of spectral functions are derived. These representations have the advantage of displaying two-body unitarity explicitly. Also, the deduction of their support properties is quite simple, as is shown in Sec. VI.

Section VII contains brief indications of the problems involved in setting up a system of integral equations for determination of amplitudes, the "crossing-unitarity mapping". A matrix N/D method, proposed in a recent paper,[5] is incorporated in a tentative form of the mapping. As was implied above, the N/D method does not in itself solve the problem of handling Regge poles, but since it goes some distance toward that goal I have shown how to employ it in a scheme which does not yet accomodate Regge poles.

In Sec. VIII the difficulty caused by Regge poles is delineated.

II. KINEMATICS, CROSSING, AND MANDELSTAM REPRESENTATION

There will be two spinless pseudoscalar mesons called π and M, with masses m_1 and m_2, respectively, $m_2 \geq m_1$. One is interested in the case where $m_2/m_1 \geq m_K/m_\pi$, with m_K and m_π being the masses of the physical K and π mesons. For simplicity the M meson does not carry hypercharge, but there is no difficulty in extending the formalism to include hypercharge so that M could be identified with the physical K.

Just as though there were hypercharge, $\pi\pi$ and MM intermediate states will be disallowed in the πM channel, the lowest inelastic threshold of the latter being $(3\pi)M$. Elastic channels $\pi\pi, MM, \pi M$ will be numbered 1, 2, 3, respectively, and the matrix of elastic and quasielastic plane-wave scattering amplitudes is written as

$$A(s,t)=\begin{pmatrix} A_1(s,t) & A_{12}(s,t) & 0 \\ A_{21}(s,t) & A_2(s,t) & 0 \\ 0 & 0 & A_3(s,t)\end{pmatrix}. \tag{2.1}$$

Time-reversal invariance is assumed, so that $A_{12} = A_{21}$.

In the center-of-momentum frame the momentum of a particle in the ith channel is q_i, where

$$q_i^2(s)=(s-4m_i^2)/4,\quad i=1,2, \tag{2.2}$$

$$q_3^2(s)=[s-(m_1+m_2)^2][s-(m_1-m_2)^2]/4s, \tag{2.3}$$

$s^{1/2}$ being the total energy in that channel. In terms of t, the square of the invariant momentum transfer, the cosine of the scattering angle for elastic scattering in channel i is

$$z_i=\cos\theta_i=1+t/2q_i^2. \tag{2.4}$$

For $\pi\pi\to MM$ the corresponding cosine may be written as

$$z_{12}=\cos\theta_{12}=(q_1^2+q_2^2+t)/2q_1q_2 \tag{2.5}$$

$$=-(q_1^2+q_2^2+u)/2q_1q_2, \tag{2.6}$$

where

$$u=\Sigma-s-t,\quad \Sigma=2(m_1^2+m_2^2). \tag{2.7}$$

Each of the four amplitudes has a partial-wave development

$$A_i(s,t)=\sum_{l=0}^{\infty}(2l+1)P_l(z_i)a_{il}(s),\quad i=1,2,12,3. \tag{2.8}$$

As in Ref. 1, unitarity conditions are stated in terms of partial waves. The phase-space factor r_i of the ith channel is defined to include a unit step function, which vanishes below the channel threshold s_i:

$$r_i(s)=\theta(s-s_i)q_i(s)/s^{1/2},\qquad s_1=4m_1^2,\quad s_2=4m_2^2,\quad s_3=(m_1+m_2)^2. \tag{2.9}$$

The unitarity condition is

$$[a(s_+)-a(s_-)]/2i=a(s_+)r(s)a(s_-)+F(s)+\Delta_L a(s),\quad s\geq s_1 \tag{2.10}$$

where the angular momentum index l is suppressed and

$$a=\begin{pmatrix} a_1 & a_{12} & 0\\ a_{12} & a_2 & 0\\ 0 & 0 & a_3\end{pmatrix},\quad F=\begin{pmatrix} F_1 & F_{12} & 0\\ F_{12} & F_2 & 0\\ 0 & 0 & F_3\end{pmatrix},\quad r=\begin{pmatrix} r_1 & 0 & 0\\ 0 & r_2 & 0\\ 0 & 0 & r_3\end{pmatrix},\quad a(s_\pm)=\lim_{\epsilon\to 0+}a(s\pm i\epsilon). \tag{2.11}$$

The absorption matrix (overlap matrix) F accounts for transitions to and from channels other than the ones treated explicitly. The lowest states appearing in F_1, F_2, and F_{12} are 4π states, while the lowest in F_3 is $(3\pi)K$. Notice that if $m_1=m_\pi$ and $m_2=m_K$, then $m_2\simeq 3.5m_1$, and F_2 and F_{12} are non-zero below the physical threshold s_2 of the amplitudes a_2 and a_{12}. The term $\Delta_L a$ in (2.10) is the discontinuity of a over its left-hand cut in the s plane. As will be shown, the element a_2 has a left-hand cut beginning at $s=s_2-s_1$, so that $\Delta_L a_2$ will appear in (2.10) for $s_1<s<s_2-s_1$ when s_2 is greater than $2s_1$, as it is in the case of $m_1=m_\pi$, $m_2=m_K$. All other elements of $\Delta_L a$ are zero in the region of interest $s>s_1$.

To discuss crossing symmetry, let us first consider those amplitudes in which all four legs correspond to the same type of particle $A_i(s,t)$, $i=1,2$. Since $s+t+u=4m_i^2$, each amplitude may be regarded as a function of any two Mandelstam variables:

$$A_i(s,t)=B_i(t,u)=C_i(u,s),\quad i=1,2. \tag{2.12}$$

Crossing symmetry is the statement that each of the functions A_i, B_i, and C_i is symmetric in its two variables. In terms of A_i alone, crossing symmetry may then be expressed by the relations

$$A_i(s,t)=A_i(t,s)=A_i(s,u),\quad i=1,2. \tag{2.13}$$

The other amplitudes may be treated similarly. For $\pi\pi\to MM$ one has

$$A_{12}(s,t)=B_{12}(t,u)=C_{12}(u,s), \tag{2.14}$$

where $s+t+u=2(m_1^2+m_2^2)$. Since the t and u channels are physically equivalent, B_{12} must be symmetric. In terms of A_{12} that means

$$A_{12}(s,t)=A_{12}(s,u). \tag{2.15}$$

For $\pi M\to\pi M$,

$$A_3(s,t)=B_3(t,u)=C_3(u,s), \tag{2.16}$$

and C_3 is symmetric due to the equivalence of s and

u channels. Hence

$$A_3(s,t)=A_3(u,t). \tag{2.17}$$

Finally, one observes that the s and t channels of $\pi\pi\to MM$ are the same physically as the t and s channels, respectively, of $\pi M\to\pi M$, so that

$$A_{12}(s,t)=A_3(t,s). \tag{2.18}$$

Now (2.17) follows from (2.15) and (2.18), so that crossing symmetry may be summarized as follows:

$$A_i(s,t)=A_i(t,s)=A_i(s,u),\quad i=1,2\ ,$$
$$s+t+u=4m_i^2, \tag{2.19}$$

$$A_{12}(s,t)=A_{12}(s,u)=A_3(t,s),$$
$$s+t+u=2(m_1^2+m_2^2). \tag{2.20}$$

The Mandelstam representations[6] of the three independent amplitudes A_1, A_2, and A_{12} have the following forms:

$$A_i(s,t)=\frac{1}{\pi^2}\int_{s_i}^{\infty}dx\int_{s_i}^{\infty}dy\rho_i(x,y)\left(\frac{1}{(x-s)(y-t)}+\frac{1}{(x-t)(y-u)}+\frac{1}{(x-u)(y-s)}\right),\quad i=1,2 \tag{2.21}$$

$$A_{12}(s,t)=A_3(t,s)=\frac{1}{\pi^2}\int_{s_1}^{\infty}dx\int_{s_3}^{\infty}dy\,\phi(x,y)\,\frac{1}{x-s}\left(\frac{1}{y-t}+\frac{1}{y-u}\right)+\frac{1}{\pi^2}\int_{s_3}^{\infty}dx\int_{s_3}^{\infty}dy\chi(x,y)\,\frac{1}{(x-t)(y-u)}. \tag{2.22}$$

The crossing conditions (2.19) and (2.20) are ensured by the symmetry of ρ_i and χ,

$$\rho_i(x,y)=\rho_i(y,x),\quad i=1,2 \tag{2.23}$$

$$\chi(x,y)=\chi(y,x). \tag{2.24}$$

The main task of this paper is to determine the spectral functions ρ_i, ϕ, and χ in terms of partial waves in such a way that the unitarity condition (2.10) and the crossing conditions (2.23) and (2.24) are satisfied.

III. FROISSART-GRIBOV PARTIAL-WAVE AMPLITUDES

The Froissart-Gribov elastic amplitudes for channels 1 and 2 are derived in the standard way.[7] With the help of Neumann's integral representation[8] of Q_l at integer l, one obtains

$$a_i(l,s)=\frac{1}{\pi q_i^2}\int_{s_i}^{\infty}dtQ_l(z_i)A_{it}(s,t),\quad i=1,2 \tag{3.1}$$

where A_{it} is the t-channel absorptive part

$$A_{it}(s,t)=[A_i(s,t_+)-A_i(s,t_-)]/2i$$
$$=\frac{1}{\pi}\int_{s_i}^{\infty}dx\rho_i(x,t)\left(\frac{1}{x-s}+\frac{1}{x-u}\right). \tag{3.2}$$

The amplitude (3.1), analytic in l, coincides with the physical partial wave $a_{li}(s)$ at even integer l; the latter is zero at odd l due to the Bose symmetry implied by the relation $A(s,t)=A(s,u)$. With account taken of (2.5) and (2.6), a calculation for A_{12} yields

$$a_{12}(l,s)=\frac{1}{\pi q_1q_2}\int_{s_3}^{\infty}dtQ_l(z_{12})A_{12t}(s,t), \tag{3.3}$$

with

$$A_{12t}(s,t)=\frac{1}{\pi}\int_{s_1}^{\infty}dx\,\frac{\phi(x,t)}{x-s}+\frac{1}{\pi}\int_{s_3}^{\infty}dx\,\frac{\chi(x,t)}{x-u}. \tag{3.4}$$

Again, $a_{12}(l,s)$ is equal to the physical partial wave at even l and the physical wave is zero at odd l.

For $\pi K\to\pi K$ there is no Bose symmetry and one must define two different l-analytic amplitudes, the positive-signature amplitude $a^{(+)}(l,s)$ which coincides with the physical wave at even l, and the negative-signature amplitude $a^{(-)}(l,s)$ which is physical at odd l. These are obtained as projections of

$$A_3^{(\pm)}(s,t(s,z))=A_3^R(s,t(s,z))\pm A_3^L(s,t(s,-z)), \tag{3.5}$$

where $A_3^R(s,t(s,z))$ is the part of $A_3(s,t(s,z))$ having a right-hand cut in the z plane and $A_3^L(s,t(s,z))$ the part with a left-hand cut. As is easily verified, the total amplitude is the even part of $A_3^{(+)}$ plus the odd part of $A_3^{(-)}$:

$$A_3(s,t(s,z))=\tfrac{1}{2}[A_3^{(+)}(s,t(s,z))+A_3^{(+)}(s,t(s,-z))]$$
$$+\tfrac{1}{2}[A_3^{(-)}(s,t(s,z))-A_3^{(-)}(s,t(s,-z))]$$
$$=A^R(s,t(s,z))+A^L(s,t(s,z)). \tag{3.6}$$

Consequently, the even (odd) Legendre projections of A_3 are those of $A_3^{(+)}$ ($A_3^{(-)}$).

With the notation $\Sigma=s+t+u=2(m_1^2+m_2^2)$, the terms in the decomposition $A_3(s,t)=A_3^R(s,t)+A_3^L(s,t)$ may be expressed as

$$A_3^R(s,t)=\frac{1}{\pi^2}\int_{s_1}^{\infty}dx\int_{s_3}^{\infty}dy\,\phi(x,y)\,\frac{1}{x-t}\times\left(\frac{1}{y-s}+\frac{1}{x+y+s-\Sigma}\right), \tag{3.7}$$

$$A_3^L(s,t)=\frac{1}{\pi^2}\int_{s_1}^{\infty}dx\int_{s_3}^{\infty}dy\,\phi(x,y)\,\frac{1}{(y-u)(x+y+s-\Sigma)}+\frac{1}{\pi^2}\int_{s_3}^{\infty}dx\int_{s_3}^{\infty}dy\,\chi(x,y)\frac{1}{(x-s)(y-u)}\,. \tag{3.8}$$

The amplitudes with signature are now obtained from (3.5), (3.7), and (3.8),

$$a_3^{(\pm)}(l,s)=\frac{1}{2\pi q_3^{\,2}}\int_{s_1}^{\infty}dt\,Q_l(z_3)A_{3t}^R(s,t)\pm\frac{1}{2\pi q_3^{\,2}}\int_{s_3}^{\infty}du\,Q_l(-z_3)A_{3u}^L(s,t)\,, \tag{3.9}$$

where

$$A_{3t}^R(s,t)=\frac{1}{\pi}\int_{s_3}^{\infty}dy\,\phi(t,y)\left(\frac{1}{y-s}+\frac{1}{y-u}\right), \tag{3.10}$$

$$A_{3u}^L(s,t)=\frac{1}{\pi}\int_{s_1}^{\infty}dx\,\phi(x,u)\frac{1}{x-t}+\frac{1}{\pi}\int_{s_3}^{\infty}dx\,\chi(x,u)\frac{1}{x-s}\,,$$

$$z_3=1+t/2q_3^{\,2}=1+(\Sigma-s-u)/2q_3^{\,2}\,, \tag{3.11}$$

$$\Sigma=s+t+u=2(m_1^{\,2}+m_2^{\,2}).$$

The amplitudes $a_3^{(+)}$ and $a_3^{(-)}$ should separately obey unitarity conditions

$$[a_3^{(\pm)}(l,s_+)-a_3^{(\pm)}(l,s_-)]/2i=a_3^{(\pm)}(l,s_+)r_3(s)a_3^{(\pm)}(l,s_-)+F_3^{(\pm)}(l,s)\,,$$

where the $F_3^{(\pm)}$ are appropriate analytic functions of l to be expressed presently in terms of spectral functions.

IV. PARTIAL-WAVE EXPANSION OF SPECTRAL FUNCTIONS

As was shown in Ref. 1, the elastic part of a spectral function may be expressed by making a Watson-Sommerfeld transformation[7] of the Legendre expansion of a corresponding elastic absorptive part. The discontinuity of the absorptive part, computed from the Watson-Sommerfeld representation, is the required spectral function. The goal of this section is to do the same for quasielastic spectral functions.

For the amplitudes A_1 and A_2 the required formula is already obvious from Ref. 1. One writes

$$\rho_i(s,t)=\phi_i(s,t)+\phi_i(t,s)+v_i(s,t)\,,\quad i=1,2\,, \tag{4.1}$$

where $\phi_i(s,t)$ is quasielastic with respect to the s channel; hence $\phi_i(t,s)$ is quasielastic with respect to the t channel. The central spectral function $v_i(s,t)=v_i(t,s)$ is an input parameter, involving states which are not quasielastic in either channel, and consequently is not to be calculated in terms of the explicit two-body channels alone. The Watson-Sommerfeld expansion of $\phi_i(s,t)$ is

$$\phi_i(s,t)=\frac{1}{4i}\int_{-\epsilon}dl(2l+1)P_l(z_i)[a_i(l,s_+)r_i(s)a_i(l,s_-)+a_{ij}(l,s_+)r_j(s)a_{ji}(l,s_-)]\,,\quad i=1,2;\quad i\neq j=1,2;\quad 0<\epsilon<\tfrac{1}{2}\,. \tag{4.2}$$

The designation $-\epsilon$ indicates that the path of integration is the line $\mathrm{Re}\,l=-\epsilon$. It is not difficult to verify that $\phi_i(s,t)$ is real, if the amplitudes have the property

$$a(l,s)=a(l^*,s^*)^*\,. \tag{4.3}$$

The implication of (4.1) and (4.2) for unitarity is seen by computing the discontinuity of $a_i(l,s)$ over its right-hand s cut from (3.1). The discontinuity arises from that of $A_{ii}(s,t)$, so that by (3.1) and (3.2),

$$[a_i(l,s_+)-a_i(l,s_-)]/2i=\frac{1}{\pi q_i^{\,2}}\int_{s_i}^{\infty}dt\,Q_l(z_i)\rho_i(x,t)+\delta_{i2}\Delta_L a_2(l,s)\,,$$

$$l=0,2,4,\ldots,\quad s\geq s_1,\quad i=1,2\,. \tag{4.4}$$

The calculation is here restricted to physical l to avoid having a cut in $Q_l(z_i)$ for $z_i<-1$; for arbitrary l one first divides $a_i(l,s)$ by $q_i^{\,2l}$ to eliminate that cut.[1,2] The remaining cut of $Q_l(z_i)$ for $-1<z_i<1$ produces the left-hand cut of $a_i(l,s)$, which for $i=2$ intrudes in the region $s>s_1$ to give the second term in (4.4), if $s_2>2s_1$. Now the first term of (4.1), introduced in (4.4), gives the quasielastic contributions to the unitarity sum:

$$\frac{1}{\pi q_i^{\,2}}\int_{s_i}^{\infty}dt\,Q_l(z_i)\phi_i(s,t)=a_i(l,s_+)r_i(s)a_i(l,s_-)+a_{ij}(l,s_+)r_j(s)a_{ji}(l,s_-)\,,\quad i=1,2\,. \tag{4.5}$$

As explained in Ref. 1, Eq. (2.30) ff., this is proved by substituting (4.2) and carrying out the t integration first. The remaining terms of (4.4) give the element F_i of the overlap matrix,

$$F_i(l,s)=\frac{1}{\pi q_i^2}\int_{s_i}^{\infty}dt\,Q_l(z_i)[\phi_i(t,s)+v_i(s,t)],\qquad i=1,2. \quad (4.6)$$

The term in (4.6) due to $\phi_i(t,s)$ corresponds to crossed two-particle processes and therefore gives contributions to the absorptive part of multiperipheral type. The term $v_i(s,t)$ represents particle production from collisions of a more central type, entailing exchange of at least four particles. The support of F is discussed in Sec. VI.

To construct the remaining spectral functions on similar lines, it is helpful to look at the discontinuities of a_{12} and $a_3^{(\pm)}$ simultaneously. By (3.3), (3.4) and (3.9)–(3.11) one finds that

$$[a_{12}(l,s_+)-a_{12}(l,s_-)]/2i=\frac{1}{\pi q_1q_2}\int_{s_3}^{\infty}dt\,Q_l(z_{12})\phi(s,t),\quad s>s_2 \quad (4.7)$$

$$[a_3^{(\pm)}(l,s_+)-a_3^{(\pm)}(l,s_-)]/2i=\frac{1}{2\pi q_3^2}\int_{s_1}^{\infty}dt\,Q_l(z_3)\phi(t,s)\pm\frac{1}{2\pi q_3^2}\int_{s_3}^{\infty}du\,Q_l(-z_3)\chi(s,u). \quad (4.8)$$

From (4.7) and the observations of the previous paragraph, it is clear that for $s>s_2$, one term in $\phi(s,t)$ should be

$$\phi_{12}(s,t)=\frac{1}{4i}\int_{-\epsilon}dl(2l+1)P_l(z_{12})[a_1(l,s_+)r_1(s)a_{12}(l,s_-)+a_{12}(l,s_+)r_2(s)a_2(l,s_-)], \quad (4.9)$$

which gives the quasielastic part of (4.7) for $s>s_2$,

$$\frac{1}{\pi q_1q_2}\int_{s_3}^{\infty}dt\,Q_l(z_{12})\phi_{12}(s,t)=a_1(l,s_+)r_1(s)a_{12}(l,s_-)+a_{12}(l,s_+)r_2(s)a_2(l,s_-). \quad (4.10)$$

The integral in (4.9) will not make sense for $s_1<s<s_2$ because $a_{12}(l,s_-)$ acquires a factor $\exp(-i\pi l/2)$ and thus blows up exponentially as $\mathrm{Im}\,l\to+\infty$. This factor arises because $Q_l(z_{12})$ behaves essentially as $[z_{12}+(z_{12}^2+1)^{1/2}]^{-l-1}$ and z_{12} is imaginary for $s_1<s<s_2$. Nevertheless, a Watson-Sommerfeld representation of ϕ_{12} may be stated in terms of a modified amplitude $\hat{a}_{12}(l,s_-)$, as is shown in Sec. V. The second term in ϕ_2 of (4.2) does not put convergence of the integral in jeopardy, because for $s<s_2$ the exponents of the two exponential factors have opposite signs.

Suppose that the definition of ϕ_{12} is completed as in Sec. V. In analogy to (4.1) one may write

$$\phi(s,t)=\phi_{12}(s,t)+\phi_3(t,s)+u(s,t), \quad (4.11)$$

$$\chi(s,u)=\chi_3(s,u)+\chi_3(u,s)+w(s,u), \quad (4.12)$$

where u and w are central spectral functions analogous to v_i. Since χ is symmetrical,

$$w(s,u)=w(u,s). \quad (4.13)$$

The elastic part of (4.8) will come from $\phi_3(s,t)$ and $\chi_3(s,u)$, whereas the other terms in (4.11) and (4.12) contribute to the overlap function $F_3^{(\pm)}(l,s)$. The correct elastic part as stated in (3.12) is achieved with the following choices:

$$\phi_3(s,t)=\frac{1}{4i}\int_{-\epsilon}dl(2l+1)P_l(z_3)r_3(s)[a_3^{(+)}(l,s_+)a_3^{(+)}(l,s_-)+a_3^{(-)}(l,s_+)a_3^{(-)}(l,s_-)], \quad (4.14)$$

$$\chi_3(s,u)=\frac{1}{4i}\int_{-\epsilon}dl(2l+1)P_l(-z_3)r_3(s)[a_3^{(+)}(l,s_+)a_3^{(+)}(l,s_-)-a_3^{(-)}(l,s_+)a_3^{(-)}(l,s_-)], \quad (4.15)$$

In (4.14), z_3 is to be determined as a function of s and t, whereas in (4.15) it is taken as a function of s and u, in accord with the situation of (4.8). When (4.11), (4.12), (4.14), and (4.15) are substituted in (4.8), a calculation analogous to that of Ref. 1, Eq. (2.30) ff., yields the unitarity condition (3.12) for $a_3^{(\pm)}$ with overlap function

$$F_3^{(\pm)}(l,s)=\frac{1}{2\pi q_3^2}\int_{s_1}^{\infty}dt\,Q_l(z_3)[\phi_{12}(t,s)+u(t,s)]\pm\frac{1}{2\pi q_3}\int_{s_3}^{\infty}du\,Q_l(-z_3)[\chi_3(u,s)+w(s,u)]. \quad (4.16)$$

The corresponding overlap function of a_{12} is

$$F_{12}(l, s)=\frac{1}{\pi q_1 q_2}\int_{s_3}^{\infty} dt\, Q_l(z_{12})[\phi_3(t, s)+u(s, t)], \quad s>s_2 . \tag{4.17}$$

Note that the formula (4.17) does not hold for $s_1 < s < s_2$.

V. EXTENDED UNITARITY AND ROTATED FROISSART-GRIBOV PROJECTIONS

As was seen in the previous section, the Froissart-Gribov amplitude $a_{12}(l, s_-)$ has exponential growth at large Iml for s in the interval (s_1, s_2) and for that reason is not suitable for partial-wave expansion of the spectral function $\phi_{12}(s, t)$ when $s<s_2$. To seek a remedy, it is useful to examine the plane-wave form of unitarity for $A_{12}(s, t)$

$$A_{12s}(s_+, t_{12}(s,\hat n_1\cdot\hat n_2)) = F_{12}(s, t_{12}(s,\hat n_1\cdot\hat n_2))+\frac{r_1(s)}{4\pi}\int d\Omega_{\hat n} A_1(s_+, t_1(s,\hat n_1\cdot\hat n))A_{12}(s_-, t_{12}(s,\hat n\cdot\hat n_2))$$
$$+\frac{r_2(s)}{4\pi}\int d\Omega_{\hat n} A_{12}(s_+, t_{12}(s,\hat n_1\cdot\hat n))A_2(s_-, t_2(s,\hat n\cdot\hat n_2)) . \tag{5.1}$$

Here $\hat n_1$ ($\hat n_2$) is a unit vector in the direction of the π (M) momentum in the center-of-momentum frame, and the integrations are over the solid angle associated with intermediate momentum direction $\hat n$. The momentum transfers have been given channel labels; by (2.4) and (2.5),

$$t_i(s,z)=2q_i^2(z_i-1), \quad i=1,2 \tag{5.2}$$

$$t_{12}(s,z)=2q_1q_2z_{12}-q_1^2-q_2^2 . \tag{5.3}$$

It is usually assumed in applications of S-matrix theory that "extended unitarity" holds.[9-13] In the present example, extended unitarity means that (5.1) is replaced by the following when $s_1<s<s_2$:

$$A_{12s}(s_+, 2i q_1\tilde q_2\hat n_1\cdot\hat n_2-q_1^2+\tilde q_2^2)=F_{12}(s, 2iq_1\tilde q_2\hat n_1\cdot\hat n_2-q_1^2+\tilde q_2^2)$$
$$+\frac{r_1(s)}{4\pi}\int d\Omega_{\hat n} A_1[s_+, 2q_1^2(\hat n_1\cdot\hat n-1)]A_{12}(s_-, 2i q_1\tilde q_2\hat n\cdot\hat n_2-q_1^2+\tilde q_2^2) , \tag{5.4}$$

where

$$\tilde q_2(s)=\left(\frac{4m_2^2-s}{4}\right)^{1/2} . \tag{5.5}$$

The overlap term F_{12} is zero for $s<(4m_1)^2$. Notice that the procedure in passing from (5.1) to (5.4) is not an analytic continuation in s; the momentum q_2 is replaced by its analytic continuation $i\tilde q_2$, but the product $A_1(s_+,\cdot)\,A_{12}(s_-,\cdot)$ is not an analytic continuation of the corresponding function from above threshold, and the term involving MM intermediate states has been simply dropped.

The branch of the square root chosen in (5.5) is immaterial. If q_2 were replaced by $-i\tilde q_2$ instead of by $i\tilde q_2$, the same statement (5.4) would be obtained, as is seen by making the change of variable $\hat n_2\to-\hat n_2$. More generally, one can multiply $\hat n_2$ by a parameter λ and do an analytic continuation in λ beginning at $\lambda=1$, since each of the functions $A_{12s}(s_+, t)$, $A_{12}(s_-, t)$, and $F_{12}(s, t)$ is analytic in t. At $\lambda=1$, the partial-wave projection of $A_{12}(s_-, t)$ entails an integration along a complex line in the t plane, rather than the usual integration on the real t axis,

$$\tfrac12\int_{-1}^{1} dz\, P_l(z)A_{12}(s_-, 2iq_1\tilde q_2 z-q_1^2+\tilde q_2^2) . \tag{5.6}$$

By introduction of the Mandelstam representation (2.22) and reversal of integration order in the usual manner, one sees that at physical l, (5.6) is equal to the Froissart-Gribov amplitude (3.5), continued from $s-i\epsilon$, $s>s_2$, to $s-i\epsilon$, $s_1<s<s_2$:

$$a_{12}(l, s_-)=\tfrac12\int_{-1}^{1} dz\, P_l(z)A_{12}(s_-, 2i q_1\tilde q_2 z-q_1^2+\tilde q_2^2)$$
$$=\frac{1}{-i\pi q_1\tilde q_2}\int_{s_3}^{\infty} dt\, Q_l\left(\frac{q_1^2-\tilde q_2^2+t}{-2i q_1\tilde q_2}\right)\times A_{12t}(s, t),$$
$$l=0, 2, 4, \ldots . \tag{5.7}$$

Here q_2 is defined to be positive for $s>s_2$, with its cut on $(-\infty, s_2)$, whereas $\tilde q_2$ is positive for $s<s_2$ with cut $(s_2, +\infty)$. Now $Q_l(z)$ has the representation (Ref. 8, Sec. 3.7, Eq. 12)

$$Q_l(z)=\int_0^{\infty} du\,[z+(z^2-1)^{1/2}\cosh u]^{-l-1} \tag{5.8}$$

so that

$$Q_l(iz) = e^{-i\pi(l+1)/2} \times \int_0^\infty du\,[z + (z^2+1)^{1/2}\cosh u]^{-l-1} . \tag{5.9}$$

With real z, the integral in (5.9) clearly cannot counteract the first factor at large $\mathrm{Im}\, l$, so that $Q_l(iz)$ has exponential growth at large $\mathrm{Im} l$. The partial wave a_{12} inherits exponential growth from Q_l in (5.7).

The difficulty clearly has to do with the fact that the partial-wave projection was in a complex direction in the t plane. By continuing in the parameter λ defined above, and then taking the projection, one can avoid the difficulty. Passing from $\lambda = 1$ to $\lambda = -i$, one obtains a "rotated" Froissart-Gribov amplitude corresponding to integration on the real t axis,

$$\hat{a}_{12}(l, s_-) = \tfrac{1}{2}\int_{-1}^{1} dz\, P_l(z) A_{12}(s_-, 2q_1\tilde{q}_2 z - q_1^2 + \tilde{q}_2^2)$$
$$= \frac{1}{\pi q_1 \tilde{q}_2}\int_{s_3}^{\infty} dt\, Q_l\left(\frac{q_1^2 - \tilde{q}_2^2 + t}{2q_1\tilde{q}_2}\right) A_{12t}(s,t) . \tag{5.10}$$

Not only is $\hat{a}_{12}$ well behaved at large $\mathrm{Im} l$, it is also sufficient to express extended unitarity. Since

$$P_l(\hat{n}_1 \cdot \hat{n}_2) = \left(\frac{4\pi}{2l+1}\right)^{1/2} \sum_{m=-l}^{l} Y_{lm}(\hat{n}_1) Y^*_{lm}(\hat{n}_2) , \tag{5.11}$$

where the spherical harmonics Y_{lm} are orthogonal on the unit sphere, Legendre expansion of each amplitude in (5.4) with $\lambda = -i$ yields

$$[\hat{a}_{12}(l, s_+) - \hat{a}_{12}(l, s_-)]/2i = a_1(l, s_+) r_1(s) \hat{a}_{12}(l, s_-) + \hat{F}_{12}(l, s), \quad l = 0, 2, 4, \ldots \tag{5.12}$$

where

$$\hat{F}_{12}(l, s) = \tfrac{1}{2}\int_{-1}^{1} dz\, P_l(z) F_{12}(s, 2q_1\tilde{q}_2 z - q_1^2 + \tilde{q}_2^2) . \tag{5.13}$$

The equations (5.12) are equivalent to (5.4) and constitute a statement of extended unitarity for A_{12}.

To derive a Watson-Sommerfield representation of ϕ_{12} for $s < s_2$, note that the contribution of the 2π intermediate state to A_{12s} is

$$A_{12s}^{(2\pi)}(s,t) = \sum_{l=0}^{\infty} (2l+1) P_l^{(e)}\left(\frac{q_1^2 - \tilde{q}_2^2 + t}{2q_1\tilde{q}_2}\right) \times a_1(l, s_+) r_1(s) \hat{a}_{12}(l, s_-) , \tag{5.14}$$

where $P_l^{(e)}(z)$ is the even part of $P_l(z)$. A Watson-Sommerfeld transformation yields

$$A_{12s}^{(2\pi)}(s,t) = \frac{i}{2}\int_{-\epsilon} dl\, \frac{2l+1}{\sin\pi l} P_l^{(e)}\left(\frac{q_1^2 - \tilde{q}_2^2 + t}{2q_1\tilde{q}_2}\right) \times a_1(l, s_+) r_1(s) \hat{a}_{12}(l, s_-) . \tag{5.15}$$

The discontinuity of (5.15) over the t cut is the required spectral function; it arises from the discontinuity of $P_l(-z)$ for $z > 1$,[14]

$$[P_l(-z_+) - P_l(-z_-)]/2i = -\sin\pi l\, P_l(z) . \tag{5.16}$$

The resulting formula is

$$\phi_{12}(s,t) = \frac{1}{4i}\int_{-\epsilon} dl\,(2l+1) P_l\left(\frac{q_1^2 - \tilde{q}_2^2 + t}{2q_2\tilde{q}_2}\right) a_1(l, s_+) r_1(s) \hat{a}_{12}(l, s_-), \quad s_1 < s < s_2 . \tag{5.17}$$

An expression for the rotated overlap function $\hat{F}_{12}$ of (5.13) may now be deduced from (5.10). For even integer l the latter gives

$$[\hat{a}_{12}(l, s_+) - \hat{a}_{12}(l, s_-)]/2i = \frac{1}{\pi q_1\tilde{q}_2}\int_{s_3}^{\infty} dt\, Q_l\left(\frac{q_1^2 - \tilde{q}_2^2 + t}{2q_1\tilde{q}_2}\right)[\phi_{12}(s,t) + \phi_3(t,s) + u(s,t)] . \tag{5.18}$$

The contribution of ϕ_{12} to (5.18) for $s < s_2$, evaluated by introducing (5.17) and reversing the order of t and l integrations, is $a_{1+} r_1 \hat{a}_{12-}$. Thus,

$$\hat{F}_{12}(l, s) = \frac{1}{\pi q_1\tilde{q}_2}\int_{s_3}^{\infty} dt\, Q_l\, \frac{q_1^2 - \tilde{q}_2^2 + t}{2q_1\tilde{q}_2}\, [\phi_3(t,s) + u(s,t)], \quad s_1 < s < s_2 . \tag{5.19}$$

Of course, extended unitarity may be stated as well in terms of the ordinary amplitude a_{12}, even if the spectral function ϕ_{12} cannot be expressed simply in terms of a_{12}. For $s < s_2$, the overlap function associated with a_{12} is

$$F_{12}(l, s) = \frac{1}{\pi q_1 q_2} \int_{s_3}^{\infty} dt\, Q_l\left(\frac{q_1^2 + q_2^2 + t}{2q_1 q_2}\right)\phi(s, t) - a_1(l, s_+) r_1(s) a_{12}(l, s_-) \ . \qquad (5.20)$$

Unfortunately, F_{12} cannot be given a simple form analogous to expression (5.19) for $\hat{F}_{12}$ when $s < s_2$. The mismatch between the argument of Q_l in (5.20) and that of P_l in (5.17) rules out a calculation of the sort that led from (5.18) to (5.19).

VI. SUPPORT OF SPECTRAL FUNCTIONS

The support of each of the quasielastic spectral functions ϕ_1, ϕ_2, ϕ_{12}, ϕ_3, and χ_3 is determined by the behavior of its constituent partial waves as functions of l at large Rel. That behavior in turn depends directly on the range of forces, which is to say on the masses of particles exchanged. The computations of support via partial waves thus have a clear physical interpretation, and in my opinion are a good deal more understandable than the corresponding computations based on the Mandelstam plane-wave unitarity condition on spectral functions.[6,11,12]

Let us first review the computation of support in the simplest case with one explicit channel; cf. Ref. 1, Eqs. (2.20)–(2.26). The Froissart-Gribov representation,

$$a(l, s) = \frac{1}{\pi q^2} \int_{s_0}^{\infty} dt\, Q_l(1 + t/2q^2) A_t(s, t) \ , \qquad (6.1)$$

leads to the bound

$$|a(l, s_\pm)| \leq \kappa(|l| + 1)^{-3/2}[z_0 + (z_0^2 - 1)^{1/2}]^{-\mathrm{Re}\,l} \ , \quad z_0 = 1 + s_0/2q^2 \ , \quad \mathrm{Re}\,l \geq -\epsilon \qquad (6.2)$$

where κ is independent of s and l. The presence of the factor $(|l| + 1)^{-3/2}$, established by means of partial integration [Ref. 2, Eq. (3.24) ff.], depends on certain smoothness properties of $A_t(s, t)$. The required smoothness has been established, in the sense that an existence theorem for a solution of the crossing-unitarity equations which implies that smoothness has been proved.[15] As is well known, the Legendre function P_l is entire in l and has the bound

$$|P_l(z)| \leq \kappa[z + (z^2 - 1)^{1/2}]^{\mathrm{Re}\,l} \ , \quad \mathrm{Re}\,l > -\tfrac{1}{2} \ . \qquad (6.3)$$

Since $a(l, s_\pm)$ is analytic in l for Re$l > -\epsilon$, the elastic spectral function

$$\phi(s, t) = \frac{1}{4i} \int_{-\epsilon} dl\, (2l + 1) P_l(z) r(s) a(l, s_+) a(l, s_-) \qquad (6.4)$$

may be treated as a contour integral over an analytic function. In view of (6.2) and (6.3), the contour may be closed by an infinite semi-circle in the right-hand half plane provided that[16]

$$[z_0 + (z_0^2 - 1)^{1/2}]^2 \geq z + (z^2 - 1)^{1/2} \ . \qquad (6.5)$$

Since the integrand is analytic for Re$l > -\epsilon$, $\phi(s, t)$ is zero when (6.5) holds. When equality holds in (5.5), a computation shows that

$$z = 2z_0^2 - 1 \ , \qquad (6.6)$$

which is to say that

$$t = 4s_0 + \frac{s_0^2}{q^2} = \frac{16m^2 s}{s - 4m^2} \ , \qquad (6.7)$$

where $s_0 = 4m^2$. Thus the support of $\phi(s, t)$ is the region for which

$$t > \frac{16m^2 s}{s - 4m^2} \ , \qquad (6.8)$$

(see Fig. 1). The asymptotes of the boundary curve, $s = 4m^2$ and $t = 16m^2$, are easily understood. Since $\phi(s, t)$ represents elastic "rescattering" in the s channel, $\pi\pi \to \pi\pi \to \pi\pi$, it must vanish below threshold, $s < 4m^2$. Since elastic rescattering of pseudoscalar mesons requires the exchange of at least four mesons, two for each single scattering, $\phi(s, t)$ must be zero for $t < (4m)^2$.

The exponential factor in (6.2) may be interpreted heuristically in terms of a quasiclassical picture. The amplitude for scattering of a particle with momentum q and impact parameter b by a Yukawa potential of range $1/2M$ should be proportional to

$$e^{-2Mb} = e^{-2Ml/q} \ , \qquad (6.9)$$

where $l = bq$ is the angular momentum. At large q, where it makes some sense to ascribe a classical trajectory to the particle,

$$\ln[z_0 + (z_0^2 - 1)^{1/2}] \sim 2m/q \ . \qquad (6.10)$$

If the Yukawa potential arises from the exchange of two mesons with mass m each, then $M = m$ and (6.2) and (6.9) agree.

Returning to the coupled-channel problem, I shall calculate the exponential factors analogous to that of (6.2) and not mention the inverse power of $|l|$ which is invoked to secure convergence of the integral over the line Re$l = -\epsilon$. Clearly, $|l|^{-3/2}$ is a stronger rate of decrease than necessary. Presumably, a sufficient rate of decrease will emerge from a full analysis of the coupled-chan-

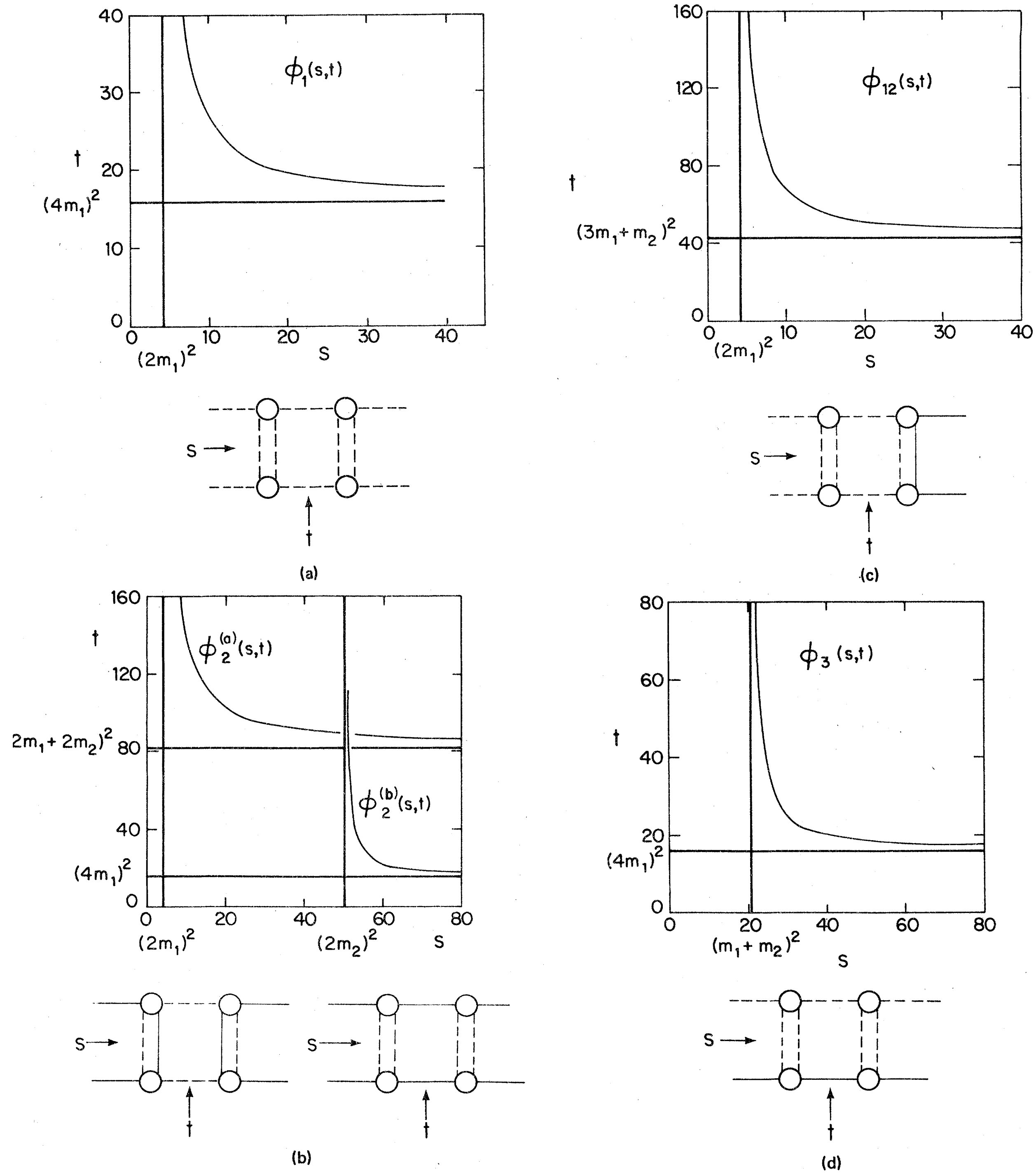
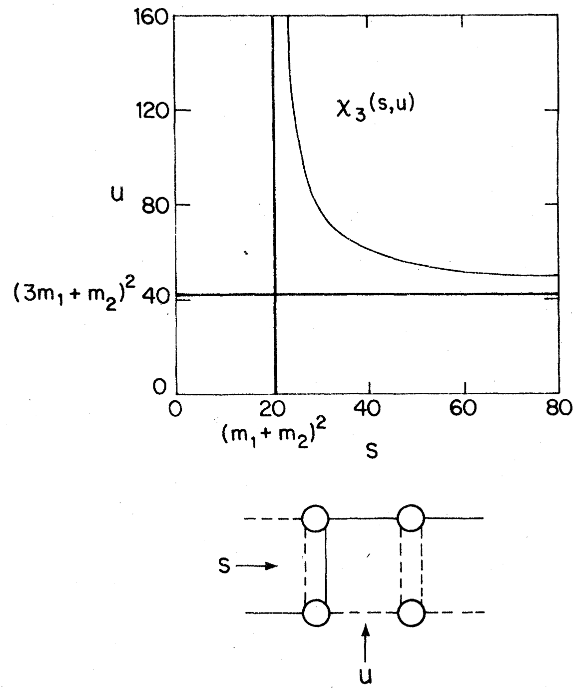
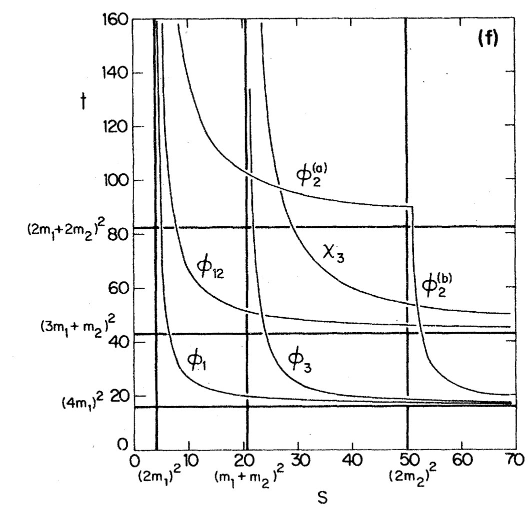


FIG. 1. Support boundary of the quasielastic spectral function: (a) $\phi_1(s,t)$ for $\pi\pi\rightarrow\pi\pi$, and corresponding diagram showing states of lowest mass in s and t channels. Dashed lines are π mesons. (b) $\phi_2(s,t)=\phi_2^{(a)}(s,t)+\phi_2^{(b)}(s,t)$ for $MM\rightarrow MM$, and corresponding diagrams for $\phi_2^{(a)}$ and $\phi_2^{(b)}$ showing states of lowest mass in s and t channels. Dashed (solid) lines are $\pi(M)$ mesons and $m_2/m_1=3.54$. (c) $\phi_{12}(s,t)$ for $\pi\pi\rightarrow MM$, and corresponding diagram showing states of lowest mass in s and t channels. Dashed (solid) lines are $\pi(M)$ mesons and $m_2/m_1=3.54$. (d) $\phi_3(s,t)$ for $\pi M\rightarrow\pi M$, and corresponding diagram showing states of lowest mass in s and t channels. Dashed (solid) lines are $\pi(M)$ mesons and $m_2/m_1=3.54$. (e) $\chi_3(s,u)$ for $\pi M\rightarrow\pi M$, and corresponding diagram showing states of lowest mass in s and u channels. Dashed (solid) lines are $\pi(M)$ mesons and $m_2/m_1=3.54$. (f) Support boundaries of the complete set of quasielastic spectral functions, all plotted on the same scale, with $m_2/m_1=3.54$.



(e)



(f)

FIG. 1. (*Continued.*)

nel equations. The following bounds are obtained from (3.1), (3.3), and (5.10):

$$|a_i(l,s_\pm)| \leq \kappa[\zeta_i+(\zeta_i^2-1)^{1/2}]^{-\mathrm{Re}\,l},$$
$$\zeta_i = 1+s_i/2q_i^2, \quad i=1,2 \qquad (6.11)$$

$$|a_{12}(l,s_+)| \leq \kappa[\zeta_{12}+(\zeta_{12}^2-1)^{1/2}]^{-\mathrm{Re}\,l},$$
$$\zeta_{12}=(q_i^2+q_2^2+s_3)/2q_1q_2, \quad s>s_2 . \qquad (6.12)$$

$$|\hat{a}_{12}(l,s_+)| \leq \kappa[\hat{\zeta}_{12}+(\hat{\zeta}_{12}^2-1)^{1/2}]^{-\mathrm{Re}\,l},$$
$$\hat{\zeta}_{12}=(q_1^2-\tilde{q}_2^2+s_3)/2q_1\tilde{q}_2, \quad s_1<s<s_2 . \qquad (6.13)$$

In the case of $a_3^{(\pm)}$, one needs separate bounds on the two terms in (3.9) that make it up. Write

$$a_3^{(\pm)} = b \pm c . \qquad (6.14)$$

Then, according to (4.14) and (4.15), the amplitude combinations in $\phi_3(s,t)$ and $\chi_3(s,u)$ are, respectively,

$$d(l,s)=b(l,s_+)b(l,s_-)+c(l,s_+)c(l,s_-) , \qquad (6.15)$$

$$f(l,s)=b(l,s_+)c(l,s_-)+b(l,s_-)c(l,s_+) . \qquad (6.16)$$

The exponential bounds of these combinations are

$$|d(l,s)| \leq \kappa[\zeta_3+(\zeta_3^2-1)^{1/2}]^{-2\,\mathrm{Re}\,l},$$
$$\zeta_3 = 1+s_1/2q_3^2 , \qquad (6.17)$$

$$|f(l,s)| \leq \kappa[\zeta_3+(\zeta_3^2-1)^{1/2}]^{-\mathrm{Re}\,l}$$
$$\times[\zeta_4+(\zeta_4^2-1)^{1/2}]^{-\mathrm{Re}\,l},$$
$$\zeta_4=(s+s_3)/(s-s_3) = -z_3(s,s_3) . \qquad (6.18)$$

The spectral function boundaries are deduced in analogy with (6.5) as follows. The results are represented graphically in Figs. 1(a)–1(f), with m_2/m_1 equal to the physical mass ratio $m_K/m_\pi = 3.54$.

(i) $\phi_1(s,t)$

The support of ϕ_1 is determined by the first term in (4.2) and the computation is exactly the same as that of the illustration above. Thus the boundary curve is

$$t=(4m_1)^2+\frac{64\,m_1^4}{s-4m_1^2} . \qquad (6.19)$$

(ii) $\phi_2(s,t)$

For ϕ_2, both terms in (4.2) must be considered, since their support boundaries intersect. The two boundary equations are

$$[\zeta_{12}+(\zeta_{12}^2-1)^{1/2}]^2 = z_2+(z_2^2-1)^{1/2} , \qquad (6.20)$$

$$[\zeta_2+(\zeta_2^2-1)^{1/2}]^2 = z_2+(z_2^2-1)^{1/2} , \qquad (6.21)$$

which are equivalent to

$$t=(2m_1+2m_2)^2+16m_1^2(m_1+m_2)^2/(s-4m_1^2) , \qquad (6.22)$$

$$t=(4m_1)^2+64m_1^4/(s-4m_2^2) . \qquad (6.23)$$

(iii) $\phi_{12}(s,t)$

For $s>s_2$, the support boundary of ϕ_{12}, determined by the first term in (4.9) is

$$[\zeta_1+(\zeta_1^2-1)^{1/2})][\zeta_{12}+(\zeta_{12}^2-1)^{1/2}] = z_{12}+(z_{12}^2-1)^{1/2}, \quad (6.24)$$

which is to say

$$t=(3m_1+m_2)^2+\frac{32m_1^3(m_1+m_2)}{s-4m_1^2}. \quad (6.25)$$

The same formula (6.25) is obtained for $s_1 < s < s_2$. By (6.13) and (5.16) the boundary equation is

$$[\zeta_1+(\zeta_1^2-1)^{1/2}][\hat{\zeta}_{12}+(\hat{\zeta}_{12}^2-1)^{1/2}] = \hat{z}_{12}+(\hat{z}_{12}^2-1)^{1/2}, \quad (6.26)$$

where

$$\hat{z}_{12}=(q_1^2-\tilde{q}_2^2+t)/2q_1\tilde{q}_2. \quad (6.27)$$

Since a factor $\tilde{q}_2^{-1}$ cancels in (6.26), and since $q_2^2=-\tilde{q}_2^2$, Eq. (6.26) actually defines the same relation between t and s as is defined by (6.24).

(iv) $\phi_3(s,t)$

According to (6.15) and (6.17), the support of ϕ_3 is bounded by the curve

$$[\zeta_3+(\zeta_3^2-1)^{1/2}]^2=z_3+(z_3^2-1)^{1/2}, \quad (6.28)$$

which is equivalent to

$$t=(4m_1)^2+\frac{64sm_1^4}{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}. \quad (6.29)$$

(v) $\chi_3(s,u)$

By (4.15), (6.16), and (6.18) the support boundary of χ_3 is given by

$$[\zeta_3+(\zeta_3^2-1)^{1/2}][\zeta_4+(\zeta_4^2-1)^{1/2}]=-z_3+(z_3^2-1)^{1/2}, \quad (6.30)$$

which is equivalent to

$$u=(3m_1+m_2)^2+\frac{16m_1^2(m_1+m_2)^2}{s-(m_1+m_2)^2}. \quad (6.31)$$

In each of the cases (i)–(v), the asymptotes of the boundary curves have a simple interpretation in terms of the particle states in direct and crossed channels, which is illustrated in the Feynman like diagrams of Figs. 1(a)–1(f).

The supports of the overlap functions are determined by those of the quasielastic spectral functions. By (4.6) and (6.19), (6.22) and (6.23), one sees that

$$F_1(l,s)=F_2(l,s), \quad (2m_1)^2 < s < (4m_1)^2. \quad (6.32)$$

Similarly, by (4.16), (6.25), and (6.31),

$$F_3^{(\pm)}(l,s)=0, \quad (m_1+m_2)^2 < s < (3m_1+m_2)^2. \quad (6.33)$$

Finally, (5.18) and (6.29) show that

$$F_{12}(l,s)=0, \quad (2m_1)^2 < s < (4m_1)^2. \quad (6.34)$$

VII. THE CROSSING-UNITARITY MAPPING

The above work specifies the form of spectral functions required for crossing symmetry and quasielastic unitarity. It remains to formulate a system of integral equations by which crossing-symmetric unitary amplitudes might be determined; i.e., an appropriate crossing-unitarity mapping.[1,2] A formulation based on the N/D method, analogous to that of Refs. 1 and 2, meets certain technical complications not present in the case of one explicit channel. Since the problems involved are not completely settled, I shall confine myself here to a brief review of the main points.

(i) As has been seen above, crossing symmetry requires inelastic effects beyond those associated with states included explicitly in the direct channel. At the least, one must have the crossed two-particle terms which through (4.6), (4.16), (4.17), (5.18), and (5.19) give contributions to the overlap functions. Hence a coupled-channel N/D scheme with crossing symmetry requires that $F(l,s)$ along with the usual left-hand cut term $B_L(l,s)$, be an input in the N/D integral equation. An appropriate integral equation for the purpose, a revision of one proposed earlier,[17] is described in Ref. 5. The equation is nonlinear and marginally singular, so that its application is a bit more complicated than that of the corresponding single-channel equation. On the other hand, the overlapping left-hand and right-hand cuts of $a_2(l,s)$ present no difficulty; the N/D method handles that situation quite automatically. The reason can be understood more easily in the matrix N/D method without absorption,[18] but the result is true as well in the presence of absorption.[5]

(ii) As is discussed in Ref. 5, $F_{12}(l,s)$ is not real, in general, at real l; the matrix $F(l,s)$ as a whole is merely Hermitian. The spectral function ϕ_{12} as given by (4.9) and (5.16) is not real in general, whereas the other quasielastic spectral functions are real by virtue of the property $a_i(l,s) = a_i^*(l^*,s^*)$ of partial waves. Since the total spectral function ϕ of (4.11) must be real, it follows that u must be complex in such a way that

$$\mathrm{Im}u(s,t)=-\mathrm{Im}\phi_{12}(s,t). \quad (7.1)$$

Hence F_{12} as given in (4.17) and (5.19) has an imaginary part due to Imu. Unlike the other central spectral functions, u is not an entirely arbitrary input; only its real part may be chosen arbitrarily.

In correspondence with (7.1), at real l one has

$$\text{Im}F_{12}(l, s) = -\text{Im}[a_1(l, s_+)r_1(s)a_{12}(l, s_-)] \quad (7.2)$$

by virtue of the unitarity condition (2.10). As was shown in Ref. 5, the fact that $\text{Im}F_{12}$ is given in terms of elastic amplitudes has the consequence that ReF and B_L determine the scattering amplitude $a(l, s)$; the problem of constructing amplitudes is over-determined if B_L and both the real and imaginary parts of F are specified. Consequently, it happens that when an amplitude $a(l, s)$ is constructed by the N/D equation of Ref. 5 with inputs B_L and F, it will have a left-hand cut part with the input value B_L, but an overlap matrix with a new value $\tilde{F} \neq F$. It should be possible to adjust ImF by an iterative procedure, in which ReF would be held constant, so as to make $\tilde{F} = F$. Since $\tilde{F} = F$ is required in a complete scheme for construction of crossing-symmetric amplitudes, such an iteration must be taken as part of the crossing-unitarity mapping analogous to the mapping G of Eq. (1.9) in Ref. 1.

(iii) The rotated Froissart-Gribov amplitude $\hat{a}_{12}(l, s)$, found to be necessary in the work of Sec. V, does not fit very naturally into the N/D scheme. It must nevertheless be involved in the crossing-unitarity mapping, and one is therefore led to a hybrid mapping in which the ordinary amplitudes $a(l, s)$ are treated by the N/D method, but $\hat{a}_{12}(l, s)$ is not.

(iv) A tentative form of the mapping, adapted to the circumstances described in (i)–(iii), is set up as follows. The basic nonlinear equation for determination of amplitudes is construed as an equation for the triple of functions $(B_L, F, \hat{a}_{12})$. Let us write it as

$$(B_L, F, \hat{a}_{12}) = G(B_L, F, \hat{a}_{12}). \quad (7.3)$$

Here $\hat{a}_{12}(l, s_-)$ is defined only on the interval $[s_1, s_2]$. The crossing-unitarity mapping G is defined in three steps:

(1) Given (B_L, F), solve the N/D equation of Ref. 5 (an equation for ImD), and from the solution construct the matrix $a^{(1)}(l, s)$ of partial-wave amplitudes.

(2) Carry out further solutions of the N/D equation with inputs

$$(B_L, \text{Re}F + i\ \text{Im}F^{(2)}), \ldots, \quad (B_L, \text{Re}F + i\ \text{Im}F^{(k)}), \ldots,$$

where

$$\text{Im}F^{(k)} = -\text{Im}(a_+^{(k-1)} r a_-^{(k-1)}), \quad (7.4)$$

$a_\pm^{(k)} = a^{(k)}(l, s_\pm)$ being the amplitude matrix obtained from the kth solution of the equation. At complex l, the designations "ReF" and "ImF" are not to be taken literally; i ImF is understood as that part of F arising from Im$u(s, t)$,[19] and the right-hand side of (7.4) is interpreted as

$$-(a_+^{(k-1)} r a_-^{(k-1)} - a_-^{(k-1)} r a_+^{(k-1)})/2i. \quad (7.5)$$

(3) In the manner of Ref. 1, determine a new triple $(B_L', F', a_{12}') = G(B_L, F, \hat{a}_{12})$ from the Froissart-Gribov formulas (3.1), (3.3), (3.9), and (5.10), evaluating the spectral functions in terms of $\hat{a}_{12-}$ and $a_\pm = a_\pm^{(\infty)}$, the latter being the limit of the iteration of step (2). In view of (7.1), Im$u(s, t)$ is always to be replaced by $-\text{Im}\phi_{12}(s, t)$.

(v) It is plausible that the operator G has a locally unique fixed point in an appropriate Banach space if the parameters on which it depends $(v_1, v_2, \text{Re}u, w)$ are suitably restricted. A fixed-point proof is possible in the case of a single explicit channel.[15] In the coupled-channel case a proof will be harder, one reason being that convergence of the iteration in step (2) must be established.

VIII. OUTLOOK FOR FURTHER WORK

As has been seen, the partial-wave representation of coupled-channel spectral functions is neat and comprehensible. On the other hand, the formulation of the crossing-unitarity mapping offered in Sec. VII is inelegant and not easily generalized to the realistic case in which Regge poles enter the right-hand half of the l plane. In Ref. 2, the N/D method was introduced expressly to handle such poles. If it were not for the disruption of the scheme caused by the rotated amplitude $\hat{a}_{12}$, the coupled-channel N/D scheme would be similarly effective in dealing with Regge poles.

The difficulty is centered on the problem of continuing the Froissart-Gribov representation (5.9) of $\hat{a}_{12}$ from large to small values of Rel. As is well known, Regge poles arise from the divergence of the t integral: $A_{12t}(s, t)$ behaves as $t^{\alpha(s)}$, because of the presence of $\phi_{12}(s', t)$ in its spectral function. In the single-channel case, the remedy was to rewrite the Froissart-Gribov representation as a partial-wave dispersion relation in the s plane. The latter could be continued to small Rel, since the offending Regge pole appeared explicitly in the elastic-unitarity term of the right-hand cut contribution. The N/D method, replacing the partial-wave dispersion relation, provided a practical means of continuation. A similar approach for $\hat{a}_{12}$ fails, because its discontinuity over the right-hand cut in the s plane is given in a form explicitly displaying Regge poles [i.e., the form (5.11)] only for $s_1 < s < s_2$. For $s > s_2$, the discontinuity is available only in the form

$$[\hat{a}_{12}(l, s_+) - \hat{a}_{12}(l, s_-)]/2i = \frac{1}{\pi q_1 \tilde{q}_2} \int_{s_3}^{\infty} dt\, Q_l\left(\frac{q_1^2 - \tilde{q}_2^2 + t}{2q_1\tilde{q}_2}\right)\phi(s, t). \quad (8.1)$$

Since the term $\phi_{12}(s,t)$ in $\phi(s,t)$ behaves as $t^{\alpha(s)}$, one is assured of convergence of (8.1) only at sufficiently large Rel. If a means of continuation of (8.1) to small Rel could be found, then one might use the approach of the inhomogeneous Hilbert problem (generalized Omnès-Muskhelishvili method)[20] to find an analytic amplitude $\hat{a}_{12}$ satisfying the linear unitarity relation (5.11) for $s_1 < s < s_2$ and having the (continued) discontinuity (8.1) for $s > s_2$. Similar problems occur in defining F_{12} for $s < s_2$ and small Rel when there are Regge poles; see Eq. (5.19).

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